Definition and Some Properties of Elementary Functions

May 25, 2016

I. Introduction

The term "elementary function" is very often mentioned in many math classes and in books, e.g. Calculus books. In fact, the vast majority of the functions that students and scientists come across are elementary functions of a real variable. However, there is a lack of a precise mathematical definition of elementary functions. Only a few authors in their textbooks, e.g. Stewart in his Calculus books try to give a description of elementary functions. Unfortunately, these descriptions are not given properly. For example, from the descriptions of elementary functions in Stewart's book, one could conclude that the function

$$f(x) = \begin{cases} \sin x & , \ x \leq 5 \\ \ln x & , \ x > 5 \end{cases}$$

is elementary!! This is simply incorrect.

Thus, this note is written to introduce a precise mathematical definition of elementary functions of a real variable. It is not claimed to be an original research article, but rather a note that could serve the students to see a proper mathematical definition of the term Elementary Function of a Real Variable. After the definition is introduced, it is easy to see that the elementary functions of a real variable possess properties that could greatly simplify the mathematical analysis needed to be done on them. Also, many problems in mathematics deal with elementary functions or even if the functions are non-elementary, very often the studying of these non-elementary functions leads to elementary functions. According to the definition of elementary functions given in this note, there are functions that are usually considered to be non-elementary, but are elementary functions according to the definition. Since the problems are much simplified when the functions involved are elementary functions, it is a good idea to have a precise mathematical definition of the elementary functions. This definition and some more properties of elementary functions are provided in this note.

The theorems and properties of elementary functions given in this note allow for problems of continuity of functions, which often arise in calculus, to be reduced to finding the set of admissible values for a given elementary function.

Then the properties of the fundamental elementary functions can be applied to finding the set of admissible values for any given elementary function, which becomes the set of points for which the elementary function is continuous. In addition, more difficult problems, such as those involving piecewise functions like the function above, may be reduced to considering continuity of several elementary functions within a restricted domain and checking for continuity using the definition at certain points.

II. Definitions

Definition 1. The following eight functions are referred to as the fundamental elementary functions of a real variable,

$$f_1(x) = c, \quad c \in \mathbb{R}, \text{ with domain } D \subseteq \mathbb{R}$$

$$f_2(x) = x, \quad \text{with domain } D \subseteq \mathbb{R}$$

$$f_3(x) = \frac{1}{x}, \qquad \text{with domain } D \subseteq \mathbb{R} \diagdown \{0\}$$

$$f_4(x) = \sqrt[n]{x}, \;\; n \in \mathbb{N}, \text{ if } \frac{n}{2} \in \mathbb{N} \text{ then domain } D \subseteq [0, +\infty), \text{ and if } \frac{n+1}{2} \in \mathbb{N} \text{ then domain } D \subseteq \mathbb{R}$$

$$f_5(x) = \sin x, \quad \text{with domain } D \subseteq \mathbb{R}$$

$$f_6(x) = e^x, \quad \text{with domain } D \subseteq \mathbb{R}$$

$$f_7(x) = \ln x, \quad \text{with domain } D \subseteq (0, +\infty)$$

$$f_8(x) = \arccos x, \quad \text{with domain } D \subseteq [-1, 1]$$

Definition 2. For any two functions (of a real variable) $f(x)$ and $g(x)$ with domains D_f, D_g and ranges R_f, R_g, respectively, the following operations are called The Fundamental Elementary Operations on Functions:

1. Addition: $\forall x \in D = D_f \bigcap D_g \neq \emptyset$, $f(x)$ and $g(x)$ are both defined and have values a and b ($a \in R_f$ and $b \in R_g$) respectively. Thus, $\forall x \in D$ there is a unique corresponding real number $c = a + b$. Hence, we define a new function

$$O_1(x) = f(x) + g(x)$$

where the domain of $O_1(x)$ is $D = D_f \bigcap D_g$, which is called the sum of $f(x)$ and $g(x)$.

2. Multiplication: In a similar fashion as in 1, we define

$$O_2(x) = f(x) \cdot g(x)$$

where the domain of $O_2(x)$ is $D = D_f \bigcap D_g$. $O_2(x)$ is called the product of $f(x)$ and $g(x)$.

3. Composition of Functions: $\forall x \in D_f$ and $R_f \subseteq D_g$, $f(x)$ is defined and has a value $f(x) = a$, $a \in R_f$. Since $x \in D_f$, $a \in R_f \subseteq D_g$, hence $a \in D_g$. Since $a \in D_g$, $g(a) = g(f(x))$ is defined and has value $g(a) = g(f(x)) = b$. Thus, we define $O_3(x)$ to be the composite function of $f(x)$ and $g(x)$ iff

$$\begin{cases} R_f & \subseteq D_g \\ O_3(x) & = g(f(x)),\ x \in D_{O_3(x)} = D_f \end{cases}$$

Definition 3. A function $F(x)$ with domain D is called an elementary function, if it can be obtained from one and the same set of fundamental elementary functions using a finite number of fundamental elementary operations in one and the same way for all $x \in D$.

Remark. If a function is elementary, then it may be written with only one formula for all $x \in D$.

Definition 4. For a function $f(x)$ defined in a domain D, a point a is called an isolated point of D, iff

1. $a \in D$ and,
2. $\exists \epsilon > 0$ such that $\forall x \in (a - \epsilon, a) \bigcup (a, a + \epsilon)$, $x \notin D$.

III. Theorems

Theorem 1. *Subtraction could be obtained from applying the fundamental elementary operations, i.e. subtraction is an elementary operation.*

Proof. Let $f(x)$ and $g(x)$ be elementary functions. Then, $f(x) - g(x) = f(x) + (-g(x)) = f(x) + (-1)(g(x))$. Thus, the theorem follows. □

Theorem 2. *Division could be obtained from applying the fundamental elementary operations, i.e. division is an elementary operation.*

Proof. Let $f(x)$ and $g(x)$ be elementary functions. Then, $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ which is the multiplication of two elementary functions. Thus, the theorem follows. □

Theorem 3. *All polynomial functions are elementary functions.*

Proof. A general polynomial function is defined as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where $n \in \mathbb{N}$, $a_n \neq 0$, $a_j \in \mathbb{R}$, $j = 0, 1, ..., n$, and $D_f \subseteq \mathbb{R}$. For any $j \in \mathbb{N}$ such that $0 \leq j \leq n$, x^j is obtained from j multiplication(s) of x and multiplication is a fundamental elementary operation. Thus x^j is an elementary function. Since $a_j \in \mathbb{R}$ $(j = 0, 1, 2, ..., n)$, a_j is an elementary function. Then multiplication (a fundamental elementary operation) of two fundamental elementary functions is an elementary function. Since addition of a finite amount of elementary functions is an elementary function, the theorem is proven. □

Theorem 4. *All rational functions are elementary functions.*

Proof. A rational function could be defined as

$$r(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomial functions with domains D_f and D_g respectively. Then the domain of $r(x)$ is $D_r = D_f \bigcap D_g \setminus \{x \in D_g | g(x) = 0\}$. Then division is obtained from applying the fundamental elementary operations by theorem 2. Since polynomial functions are elementary functions, the theorem is proven. □

Theorem 5. *All rational powers of elementary functions are elementary functions.*

Proof. Let $f(x)$ be an elementary function with D_f. Then, a rational power function may be defined as $f(x)^{\frac{m}{n}}$ for some $m, n \in \mathbb{Z}$ $(n \neq 0)$ with domain D. D must be such that $f(x)$ is non-negative $\forall x \in D$ if n is even and m is odd. Then this may be rewritten as $\sqrt[n]{(f(x))^m}$. Then $f(x)^m$ is multiplication of $f(x)$ m times so that it is an elementary function. In addition, $\sqrt[n]{x}$ is an elementary function by definition. Thus, composition of these two is an elementary function, and the theorem is proven. □

Theorem 6. *All trigonometric functions are elementary functions.*

Proof. We may define the function $\cos x$ with domain $D \subseteq \mathbb{R}$ by $\cos x = \sin\left(x + \frac{\pi}{2}\right)$. Then, $x + \frac{\pi}{2}$ is addition of two fundamental elementary operations and thus, is elementary. Then, $\sin\left(x + \frac{\pi}{2}\right)$ is the composition of an elementary function with an elementary function. Hence, $\cos x$ is elementary as was to be shown.

Then, we may obtain $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, and $\csc x = \frac{1}{\sin x}$. □

Theorem 7. *All inverse trigonometric functions are elementary functions.*

Proof. First, $\arcsin x = \frac{\pi}{2} - \arccos x$, so $\arcsin x$ is an elementary function.

To obtain $\text{arccot} x$, let $\text{arccot} x = t$ so that $x = \cot t$ where $x \in (-\infty, +\infty)$ and $t \in (0, \pi)$. Then, $\cos t = \sin t \cdot \cot t$. For $t \in (0, \pi)$, $\sin t > 0$ and $1 + \cot^2 t = \frac{1}{\sin^2 t}$. Thus, $\sin t = \frac{1}{\sqrt{1+x^2}}$. Therefore, $\cos t = \frac{x}{\sqrt{1+x^2}}$. Hence, $t = \arccos \frac{x}{\sqrt{1+x^2}}$ so that we may define $\text{arccot} x = \arccos \frac{x}{\sqrt{1+x^2}}$ where $x \in (-\infty, +\infty)$. Since this is the composition of several elementary functions, it is clearly an elementary function. Clearly $\arctan x = \frac{\pi}{2} - \text{arccot} x$, so $\arctan x$ is an elementary function as well.

Similarly, we may define $\text{arcsec} x = \arccos \frac{1}{x}$ and $\text{arccsc} x = \arcsin \frac{1}{x}$ where $x \in (-\infty, -1] \cup [1, +\infty)$, both of which are also elementary functions. □

Theorem 8. *All logarithmic function of elementary functions are elementary functions.*

Proof. $\log_{u(x)} v(x) = \frac{\ln v(x)}{\ln u(x)} = \ln v(x) \cdot \frac{1}{\ln u(x)}$ for values of x such that $u(x) \neq 1$, $u(x) > 0$, and $v(x) > 0$. Since $\ln v(x)$ is an elementary function (composition of $\ln x$ and $v(x)$) and $\frac{1}{\ln u(x)}$ is also an elementary function (composition of $\frac{1}{x}$, $\ln x$ and $u(x)$), $\ln v(x) \cdot \frac{1}{\ln u(x)} = \log_{u(x)} v(x)$ is an elementary function. □

Theorem 9. *Functions of the form $u(x)^{v(x)}$ $(u(x) > 0)$, where $u(x)$ and $v(x)$ are elementary functions, are also elementary functions..*

Proof. For all values of x such that $u(x)$ and $v(x)$ are defined and greater than 0, $u(x)^{v(x)} = e^{v(x) \cdot \ln u(x)}$. Since the operations involved are either the fundamental elementary operations or a combination of them, $u(x)^{v(x)}$ is an elementary function. □

The following statements are theorems that are very often proven in textbooks and thus will only be stated without proofs

1. All eight, $f_1(x)$ - $f_8(x)$, fundamental elementary functions are continuous everywhere in their domains except at the isolated points and are discontinuous at the isolated points.

2. The sum of two continuous functions is also continuous everywhere in its domain except at the isolated points and is discontinuous at the isolated points.

3. The product of two continuous functions is also continuous everywhere in its domain except at the isolated points and is discontinuous at the isolated points.

4. The composition of two continuous functions is also continuous everywhere in its domain except at the isolated points and is discontinuous at the isolated points.

Thus, one has the following important theorem concerning elementary functions:

Theorem 10. *All elementary functions are continuous in their domains, except at the isolated points at which they are discontinuous.*

Proof. All of the fundamental elementary functions are continuous in their domains, except at any isolated points at which they are discontinuous. Then, since all elementary functions may be written as one equation consisting of addition, multiplication, and composition of the fundamental elementary functions by definition, all elementary functions are continuous in their domains, except at the isolated points at which they are discontinuous. □

Remark. In some real analysis books, continuity is defined in a way that all functions are continuous at their isolated points.

Theorem 11. *The function*

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

is an elementary function.

Proof. Consider $\sqrt{x^2}$. Clearly, $x^2 = |x|^2$ for all $x \in \mathbb{R}$. Thus, since $|x| \geq 0$, we may take the square root and obtain $\sqrt{x^2} = \sqrt{|x|^2} = |x|$.

Then we may clearly define $|x| = \sqrt{x^2}$. Since this is the composition of two elementary functions written in one equation, it is an elementary function. □

Theorem 12. *For a function $f(x)$ defined as*

$$f(x) = \begin{cases} g(x) & , x < a \\ h(x) & , x > a \end{cases}$$

where $g(x)$ is an elementary function in $D_g = (-\infty, a)$ and $h(x)$ is an elementary function in $D_h = (a, +\infty)$. The function $f(x)$ with domain $D_f = \mathbb{R} \setminus \{a\}$ is an elementary function.

Proof. Note that

$$\frac{|x| + x}{2x} = \begin{cases} 0 & , x < 0 \\ 1 & , x > 0 \end{cases}$$

and

$$\frac{-|x| + x}{2x} = \begin{cases} 1 & , x < 0 \\ 0 & , x > 0 \end{cases}$$

Applying a small shifting in x, the function $f(x)$ could be written as

$$f(x) = g(x)\frac{x - a - |x - a|}{2(x - a)} + h(x)\frac{x - a + |x - a|}{2(x - a)}$$

with domain $D_f = \mathbb{R} \setminus \{a\}$. Since the formula to calculate the value of $f(x)$ at any point in the domain D_f of $f(x)$ consists of only generalized elementary functions and elementary operations, $f(x)$ is an elementary function. □

Remark. Of course, it is very easy to generalize the above definitions and theorems for functions of any finite number of real or complex variables. This generalization may be done by only including the five fundamental elementary functions $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_5(x)$, and $f_6(x)$, and including taking the inverse of a function as a fundamental elementary operation. However, the function $|z|$ (modulus) is not an elementary function of a complex variable z, even if $|x|$ (absolute value) is an elementary function of a real variable.